MESSENGER OF MATHEMATICS,

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Thus

$$\begin{vmatrix} (1), & (1) & , & (1) & , \ldots\ldots\ldots\ldots(1) \\ (1), & (1)+(2), & & (1) & , \ldots\ldots\ldots\ldots \\ (1), & (1) & , & (1)+(3), & \ldots\ldots\ldots\ldots \\ \ldots & \ldots & & \ldots & \ldots \\ (1) & \ldots\ldots & & , (L_1)+(L_2) & +\&c. \end{vmatrix} = (1)\,(2)\,(3)\ldots(L).$$

IV. Finally, instead of (1), (2), (3)..., we may write any algebraical symbols; which gives the solution of this question: Express any product (1) (2) (3)...(L) as a determinant formed according to the above laws, except that the terms of the diagonal are not the series of natural numbers.

Example.

$$x^5 = \begin{vmatrix} x, & x, & x, & x, & x \\ x, & 2x, & x, & x, & x \\ x, & x, & 2x, & x, & x \\ x, & 2x, & x, & 3x, & x \\ x, & x, & x, & x, & x \end{vmatrix}.$$

We may call such determinants Smithian determinants.

ON THE DEVELOPMENT OF $\left(\frac{z}{1-e^{-z}}\right)^a$ IN A SERIES.

By *M. Édouard Lucas.*

MM. Laurent and Le Paige* have recently given the development of $u^a = \left(\frac{z}{1-e^{-z}}\right)^a$ in a series of ascending powers of z. This can also be performed in the following simple manner:

Let $B_{a,p}$ be the coefficient of $\frac{z^p}{1.2.3\ldots p}$ in the development of u^a; we shall call $B_{a,p}$ the p^{th} Bernoullian number of order a.

We have $$u = e^{B_1 z},\quad u^a = e^{B_a z},$$

wherein we are to replace powers of B, and B_a by *second suffixes* and $B_{1,n}$ by the n^{th} Bernoullian number, so that

$$B_{1,0} = 1,\quad B_{1,1} = \tfrac{1}{2},\quad B_{1,2} = \tfrac{1}{6},\quad B_{1,4} = -\tfrac{1}{30}\ldots.$$

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* Laurent, *Nouvelles Annales de Mathématiques*, 1875, t. XIV. p. 355.
Le Paige, *Annales de la Société scientifique de Bruxelles*, 1876.

We have then, by definition,

$$B_{\alpha,p} = [B_1' + B_1'' + \ldots + B_1^{(\alpha)}]^p,$$

on replacing $[B_1^{(\omega)}]^n$ by $B_{1,n}$, and, more generally,

$$B_{\lambda+\mu+\nu+\ldots,p} = [B_\lambda + B_\mu + B_\nu + \ldots]^p.$$

When α is a negative integer, we obtain, on changing the sign,

$$u^{-\alpha} = \left(\frac{1-e^{-z}}{z}\right)^{\alpha};$$

whence, denoting by $C_{\alpha,p}$ the number of combinations of α things taken p together,

$$B_{-\alpha,p} = \frac{(-1)^p}{(p+1)(p+2)\ldots(p+\alpha)}$$
$$\times [\alpha^{p+\alpha} - C_{\alpha,1}(\alpha-1)^{p+\alpha} + C_{\alpha,2}(\alpha-2)^{p+\alpha} \ldots \pm C_{\alpha,\alpha-1}].$$

But

$$\Delta^{\alpha} x^m = (x+\alpha)^m - C_{\alpha,1}(x+\alpha-1)^m + C_{\alpha,2}(x+\alpha-2)^m + \ldots + (-1)^{\alpha} x^m,$$

so that $$B_{-\alpha,p} = \frac{(-1)^p \Delta^{\alpha} 0^{p+\alpha}}{(p+1)(p+2)\ldots(p+\alpha)} \quad\ldots\ldots\ldots\ldots(1).$$

As a particular case,

$$B_{-1,p} = \frac{(-1)^p}{p+1}.$$

By differentiation,

$$z\frac{d(u^{\alpha})}{dz} = \alpha(z+1)u^{\alpha} - \alpha u^{\alpha+1},$$

and, equating the coefficients of $\frac{z^{p-1}}{1.2.3\ldots(p-1)}$,

$$\alpha B_{\alpha+1,p} = (\alpha - p) B_{\alpha,p} + \alpha p B_{\alpha,p-1} \quad\ldots\ldots\ldots\ldots(2).$$

Thus, for $\alpha = 1$,

$$B_{2,p} = pB_{1,p-1} - (p-1) B_{1,p}.$$

Put $$\frac{B_{1,p}}{p} = B_p,$$

and we have the symbolical formula

$$1.2.3\ldots B_{\alpha+1,p} = p(p-1)\ldots(p-\alpha)B^{p-\alpha}(1-B)(2-B)\ldots(\alpha-B)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

in which the powers of B are to be replaced by suffixes.

Replace p by $p-1$, and we obtain

$$1.2.3\ldots\alpha B_{\alpha+1,p-1}$$
$$=(p-1)(p-2)\ldots(p-\alpha-1)B^{p-\alpha-1}(1-B)\ldots(\alpha-B)\ldots.(4).$$

Multiply (3) by $\alpha-p+1$, (4) by $p(\alpha+1)$, and add, taking account of the formula (2); we thus find

$$1.2.3\ldots\alpha(\alpha+1)B_{\alpha+2,p}$$
$$=(p-1)(p-2)\ldots(p-\alpha-1)B^{p-\alpha-1}(1-B)(2-B)\ldots(\alpha-B)$$
$$\times[p(\alpha-p+1)B+p(\alpha+1)(p-\alpha-1)].$$

Simplifying this, we reproduce the formula (3), α being changed into $\alpha+1$. This formula, true for $\alpha=1, 2\ldots$ is thus generally true: it expresses the Bernoullian numbers of order α as a linear function of α consecutive Bernoullian numbers of the first order.

Paris, *September*, 1877.

ON THE SUCCESSIVE SUMMATIONS OF $1^m+2^m+3^m\ldots+x^m$.

By *M. Édouard Lucas.*

LET $\qquad S_{1,m}(x)=1^m+2^m+3^m+\ldots+x^m,$

$$S_{p,m}(x)=S_{p-1,m}(1)+S_{p-1,m}(2)+S_{p-1,m}(3)+\ldots+S_{p-1,m}(x),$$

and suppose that

$$S_{0,0}=1,\quad S_{p,0}=\frac{x(x+1)(x+2)\ldots(x+p-1)}{1.2.3\ldots p}.$$

We have the symbolical formula

$$S_{1,m}=\frac{(x+B_1)^{m+1}-B_1^{m+1}}{m+1}\ \ldots\ldots\ldots\ldots\ldots(1),$$

in the development of which the $m+1$ powers of B_1 are to be replaced by second suffixes and $B_{1,n}$ by the n^{th} Bernoullian number, with its proper sign. Differentiating the two sides of the equation, we have, as a formula to calculate the Bernoullian numbers, the identity

$$(x+1)^m=\frac{(x+1+B_1)^{m+1}-(x+B_1)^{m+1}}{m+1},$$

or, more generally,

$$f'(x+1)=f(x+1+B_1)-f(x+B_1).$$

To calculate $S_{2,m}$, we form the table

$$1^m,$$
$$1^m + 2^m,$$
$$1^m + 2^m + 3^m,$$
$$\dots\dots\dots\dots$$
$$1^m + 2^m + 3^m + \dots + x^m,$$

and add the columns; the sum of the p^{th} column is

$$(x - p + 1)\, p^m \text{ or } (x+1)\, p^m - p^{m+1}.$$

Thus $$S_{2,m} = (x+1)\, S_{1,m} - S_{1,m+1} \dots\dots\dots\dots (2);$$

or, symbolically expressed,

$$S_{2,m} = S_1^m (x + 1 - S_1).$$

For example $$S_{2,1} = \frac{x(x+1)(x+2)}{1.2.3},$$

$$S_{2,2} = \frac{x(x+1)^2(x+2)}{12}, \quad S_{2,3} = \frac{x(x+1)(x+2)(x^2+6x+3)}{60}.$$

In general

$$S_{p+1,m} = \frac{S_1^m (x+1-S_1)(x+2-S_1)\dots(x+p-S_1)}{1.2.3\dots p} \dots\dots (3).$$

In fact, changing x into $x+1$ the first side of the formula (3) is increased by

$$S_{p,m}(x+1),$$

and the second by

$$S^m (x+2-S_1)(x+3-S_1)\dots(x+p-S_1),$$

that is, by the second side of (3), when x is replaced by $x+1$ and p by $p-1$.

The formula (1) gives by summation

$$S_{2,m} = \frac{(S_1 + B_1)^{m+1} - S_1^{\,0} B_1^{\,m+1}}{m+1},$$

and generally, after p summations,

$$S_{p+1,m} = \frac{(S_p + B_1)^{m+1} - S_p^{\,0} B_1^{\,m+1}}{m+1}.$$

Changing p into $p+1$, we deduce

$$S_{p+1,m} = \frac{(S_p + B_1 + B_1)^{m+2} - S_p^{\,0}(B_1 + B_1)^{m+2}}{(m+1)(m+2)} - \frac{S^0_{p+1} B_1^{\,m+1}}{m+1},$$

and putting

$$B_{a,p} = [B_1 + B_1 + \dots B_1^{(\alpha)}]^p,$$

we have

$$S_{p+2,m} = \frac{(S_p + B_2)^{m+2} - S_p^{\,0} B_2^{\,m+2}}{(m+1)(m+2)} - \frac{S^0_{p+1} B_1^{\,m+1}}{m+1}.$$

In general we should find in the same manner

$$S_{p+q,m} = \frac{(S_p + B_q)^{m+q} - S_p^{\,0} B_q^{\,m+q}}{(m+1)(m+2)\dots(m+q)} - \frac{S^0_{p+1} B_{q-1}^{\;\;m+q-1}}{(m+1)p\dots(m+q-1)}$$

$$- \frac{S^0_{p+2} B_{q-2}^{\;\;m+q-2}}{(m+1)\dots(m+q-2)} - \dots - \frac{S^0_{p+q-1} B_1^{\,m+1}}{m+1} \dots\dots(4).$$

Putting $p = 0$, we obtain the development of $S_{q,m}$ as a function S_0 or of x.

Paris, *September*, 1877.

CUBE ROOTS OF PRIMES TO 31 PLACES.

By *S. M. Drach, F.R.A.S.*

THE 56-place values of the cube roots of 2 and 4 given on p. 54, have reminded me that twelve years ago I calculated the values of the cube roots of the primes from 2 to 127 to 33 places. The extraction of cube roots to a number of decimal places is so troublesome that it seems desirable to publish these values, which are given in Table I. They were obtained by the usual process of extracting cube roots, and were verified by actual multiplication, the multiplication being contracted throughout to 33 places.

The quantity ε, when preceded by 31 ciphers, is the amount by which the cube of the quantity in the second column differed from the first column, that is to say, for example, by cubing the quantity

1·25992 10498 94873 16476 72106 07278 399,

retaining 33 places throughout the process, I obtained as the cube

2 – ·00000 00000 00000 00000 00000 00000 003,

and in general, cube of number in second column = first column, + ε preceded by 31 ciphers.

The product of the root by itself gave me the root of the square of the number, and Table II. contains the cube roots of the squares of the primes from 2^2 to 127^2 found in this manner.

On development in series.

Consider, in general, a function developable in a convergent series proceeding according to positive powers of the variables, and, for example, let

$$\frac{F(x)}{Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}}=a_0+a_1\frac{x}{1}+a_2\frac{x^2}{2!}+\ldots+a_n\frac{x^n}{n!}+\ldots,$$

or under a symbolic form

$$\frac{F(x)}{Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}}=e^{ax}.$$

Denote by $f(x)$ any other function whatever, and by h an increment of x; we have the symbolic formula

$$F(hf)=Ae^{ahf(x+\alpha h)}+Be^{ahf(x+\beta h)}+Ce^{ahf(x+\gamma h)},$$

in the development of which we are to replace

$$h^0f^0 \text{ by } f(x),$$

$$h^nf^n \text{ by } h^n\frac{d^nf(x)}{dx^n},$$

$$\{ahf(x+\alpha h)\}^n \text{ by } a_nh^n\frac{d^nf(x+\alpha h)}{dx^n}.$$

In fact, it is easy to see that this formula holds for $f(x)=Ge^{kx}$, whatever G and k may be, and therefore also for any function whatever ΣGe^{kx} of x.

We have, in particular, for $\dfrac{x}{e^x-1}=e^{Bx}$,

$$hf'(x)=e^{Bhf(x+h)}-e^{Bhf(x)},$$

which is Stirling's formula; and for $\dfrac{-2x}{e^x+1}=e^{Px}$, with $P_n=2(1-2^n)B_n$, we have the formula

$$-2f(x)=e^{Phf(x+h)}+e^{Phf(x)},$$

a formula due to Boole.

Let $\dfrac{2}{e^x+e^{-x}}=e^{Ex}$, E_n denoting an Eulerian number, then

$$2f(x)=e^{Ehf(x+h)}+e^{Ehf(x-h)},$$

and similarly for many other developments.

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Paris, *November*, 1877.

axis of the elliptic cylinder so as to cut off the greatest area, to glide along the horizontal plane. Let C be the centre of the ellipse, H the hinge, HP the beam pressing at P upon the semi-cylinder. Let $HC=x$, θ be the angle which the normal at P makes with the major axis, $2r$ the length of the beam to the plane, and the equations of motion are

$$\frac{d^2\theta}{dt^2}=\frac{gr\sin\theta}{k^2}-\frac{P}{mk^2}\cdot\frac{b\tan\theta}{a\sqrt{(1-e^2\sin^2\theta)}},$$

$$\frac{d^2x}{dt^2}=\frac{P}{m'}\cos\theta,$$

where

$$x=\frac{b^2}{a}\frac{\sin\theta\tan\theta}{\sqrt{(1-b^2\sin^2\theta)}}+\frac{a\cos\theta}{\sqrt{(1-b^2\sin^2\theta)}}=\frac{a\sqrt{(1-e^2\sin^2\theta)}}{\cos\theta},$$

from whence $$\frac{dx}{d\theta}=\frac{a(1-e^2\sin\theta)}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}};$$

by aid of this equation we immediately deduce the equation of *vis viva* from the equations of motion, and we obtain finally

$$t=\frac{1}{\sqrt{(2mgr)}}\int\frac{d\theta}{\cos^2\theta}\left\{\frac{mk^2\cos^4\theta(1-e^2\sin^2\theta)+m'a^2(1-e^2)^2\sin^2\theta}{(1-e^2\sin^2\theta)(\cos a-\cos\theta)}\right\}^{\frac{1}{2}},$$

which may be reduced to an algebraical form by putting $\cos\theta=u$.

(To be continued.)

ON EULERIAN NUMBERS.

By *M. Edouard Lucas.*

1. If we put

$$\sec x=1+a_2x^2+a_4x^4+a_6x^6+\ldots\ \&c.,$$

we have, on multiplying the left-hand side of this equation by $\cos x$, and the right-hand side by the series for $\cos x$, the relation

$$a_{2n}-\frac{a_{2n-2}}{2\,!}+\frac{a_{2n-4}}{4\,!}-\ldots\pm\frac{a_2}{(2n-2)\,!}\mp\frac{1}{(2n)\,!}=0.$$

From this relation Mr. Glaisher has deduced an expression for a_{2n} as a determinant of the n^{th} order (*Messenger*, vol. VI., p. 52).

The Eulerian numbers are, in absolute value, given by the formula

$$E_{2n} = (-1)^n (2n)!\, a_{2n}.$$

We thus have, changing x into xi, the symbolic formula

$$\frac{2}{e^x + e^{-x}} = e^{Ex},$$

in the development of which the exponents of E are to be replaced by suffixes, and E_0 by unity. Getting rid of the denominators, we find, for n positive, the recurring relation

$$(E+1)^n + (E-1)^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

leading to the determinant

$$E_{2n} = (-1)^n \begin{vmatrix} 1, & 1, & 0, & 0, & 0, \ldots \\ 1, & 6, & 1, & 0, & 0, \ldots \\ 1, & 15, & 15, & 1, & 0, \ldots \\ 1, & 28, & 70, & 28, & 1, \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows})^*.$$

This determinant is formed of lines of even rank and of columns of uneven rank of the arithmetical triangle.

We have also the symbolic formula

$$2\{-1^n + 3^n - 5^n + 7^n + \ldots + (4x-1)^n\} = (4x+E)^n - E^n;$$

and, in addition, the formulæ

$$\frac{1}{1^{2n}} - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \ldots = \frac{(-1)^n \pi^{2n+1} E_{2n}}{2^{n+2}(2n)!},$$

$$\int_0^\infty \frac{x^{2n}dx}{e^{\pi x} + e^{-\pi x}} = \pm \frac{E_{2n}}{2^{2n+2}}.$$

2. Eulerian numbers are integers and they are uneven. Sherk has demonstrated that they end alternately in the figures 1 and 5. These properties can be proved as follows:

We deduce from the relation (1) for p prime the congruence

$$E_{p-1} + E_{p-3} + E_{p-5} + \ldots + E_2 + E_0 \equiv 0, \text{ (mod. } p);$$

whence, denoting by A_p the sum of the first p Eulerian numbers taken with their proper signs,

$$A_{p-1} \equiv 0, \text{ (mod. } p).$$

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* This value of E_{2n} as a determinant was given by Mr. Hammond in his paper 'On the relation between Bernoulli's numbers and the Binomial coefficients,' *Proceedings of the London Mathematical Society*, vol. VII., p. 18, (1875).—ED.

The first values are given by the formulæ

$$\begin{aligned} E_2+E_0&=0,\\ E_4+6E_2+E_0&=0,\\ E_6+15E_4+15E_2+E_0&=0,\\ \dots\dots&\dots\dots \end{aligned}$$

whence, starting from E_{p-1},

$$\left.\begin{aligned} E_{p+1}+E_0&\equiv 0,\\ E_{p+3}+3E_{p+1}+3E_2+E_0&\equiv 0,\\ E_{p+5}+10E_{p+3}+5E_{p+1}+5E_4+10E_2+E_0&\equiv 0,\\ \dots\dots&\dots\dots \end{aligned}\right\}\ (\text{mod. } p).$$

The comparison of these two systems of formulæ gives successively

$$\left.\begin{aligned} E_{p+1}&\equiv E_2,\\ E_{p+3}&\equiv E_4,\\ E_{p+5}&\equiv E_6,\\ \dots&\dots \end{aligned}\right\}\ (\text{mod. } p).$$

We have, in general,

$$E_{2n}\equiv E_{2n+k(p-1)},\ (\text{mod. } p),$$

whatever value the positive integer k may have, and consequently:

Theorem. The residues of the Eulerian numbers, for any prime modulus whatever, reproduce themselves periodically in the same order, just as the residues of powers.

These considerations are applicable, in general, to the differential coefficients of a rational fraction of e^x, but under certain conditions, as in the case of

$$\frac{\phi(1)}{\phi(e^x)}.$$

When $\phi(1)$ is zero, as in the development of $\frac{1}{1-e^x}$, the theorem does not hold; the differential coefficients are no longer integers and contain in the denominators an indefinite series of prime numbers; it is so, for example, with the Bernoullian numbers.

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gives the equations I. Conversely, from the equation I, III follows at once.

Hence Thomson's theorem, in the case of an infinitely small circuit, stated in the form IV, is precisely equivalent to the Helmholtz differential equations II, and stated in the integral form III, it is precisely equivalent to the Cauchy integrals I.

University, Melbourne,
September 26, 1877.

ON THE INTERPRETATION OF A PASSAGE IN MERSENNE'S WORKS.

By *M. Edouard Lucas.*

M. Genocchi has recently called attention, *àpropos* of a paper of mine, to a passage in Mersenne's Works, from which it results that numbers of the form $2^n - 1$ are composite, except when n has the values

$$2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257, \ldots.$$

I may observe that, in order to verify by known methods the last assertion of Mersenne, viz. that $2^{257}-1$ is a prime, the whole population of the globe, calculating simultaneously, would require more than a million of millions of millions of centuries.

Numbers of the form $2^n \pm 1$ can only be prime, with the sign $-$ if the exponent be prime, and with the sign $+$ if the exponent is a power of 2; and it is known that the primes of the latter form are those for which the circumference of a circle may be geometrically divided into equal parts. We have, then, to consider the three distinct classes

$$A = 2^{4q+3} - 1, \quad B = 2^{4q+1} - 1, \quad C = 2^{2^n} + 1.$$

I may add that Fermat has given the decomposition

$$2^{37} - 1 = 223 \times 616318177,$$

and Plana has given

$$2^{41} - 1 = 13367 \times 164511353.$$

By means of a new method M. Landry has lately found the decompositions

$$2^{43} - 1 = 431 \times 9719 \times 2099863,$$

$$2^{47} - 1 = 2351 \times 4513 \times 13264529,$$

$$2^{53} - 1 = 6361 \times 69431 \times 20394401,$$

$$2^{59} - 1 = 179951 \times 3203431780337,$$

and I have myself remarked that

$$2^{73} - 1 \equiv 0 \pmod{439},$$

and proved the theorem:—

If the numbers $4q+3$ and $8q+7$ are prime, then

$$2^{4q+3} - 1 \equiv 0 \pmod{8q+7};$$

and therefore the numbers

$$2^{83}-1,\ 2^{131}-1,\ 2^{179}-1,\ 2^{191}-1,\ 2^{239}-1,\ 2^{251}-1,\ \ldots$$

are not prime.

En résumé all these results seem to indicate that Mersenne was in possession of arithmetical methods that are now lost. I shall now indicate a new method of verification for each of the forms A and B.

1°. Numbers of the form $A = 2^{4q+3} - 1$.

Form the series of numbers

$$1,\ 3,\ 7,\ 47,\ 2207,\ 4870847,\ 27325150497407,\ \ldots,$$

in which each is equal to the square of the preceding one diminished by 2, and retain the residues to modulus A; the calculation of the residues is easily performed by successive subtractions, the first ten multiples of A having been first calculated.

If no one of the $4q+3$ first residues is equal to zero, the number A is composite; if the first zero is comprised within the limits $2q+1$ and $4q+3$, the number A is prime; in fact, if $\alpha, < 2q+1,$ denotes the position of the first zero residue, the divisors of A belong to the form $2^{\alpha}k \pm 1$, and to the quadratic form $x^2 - 2y^2$.

Example. For $A = 2^7 - 1$, we have the residues

$$1,\ 3,\ 7,\ 47,\ 48,\ 16,\ 0 \pmod{127},$$

whence the number is prime.

For $A = 2^{11} - 1$ we form the residues

$$1,\ 3,\ 7,\ 47,\ 160,\ 1034,\ 620,\ -438,\ -576,\ 160;$$

and $A, = 2047,$ is not prime and the residues reproduce themselves periodically. Thus $2^{11} - 1$ is composite,

$$2^{11} - 1 = 23 \times 89.$$

2°. Numbers of the form $B = 2^{4q+1} - 1$.

Form the series of numbers r_n,

$$1,\ -1,\ 7,\ 17,\ 5983,\ \ldots,$$

such that $$r_{n+1} = 2r_n^{\,2} - 3^{2^{n-1}},$$

and take the series of residues to the modulus B. The number B is prime if the first zero residue has a position comprised between $2q$ and $4q+1$; it is composite if no one of the $4q+1$ first residues is equal to zero; and, if α, $=2q$, is the position of the first zero residue, the divisors of B belong to the linear form $2^{\alpha}k \pm 1$, combined with those of the quadratic divisors of the form $2x^2 - y^2$.

Thus $r_8 = 5983 = 193 \times 31$; therefore $2^5 - 1$ is prime.

MATHEMATICAL NOTES.

A Problem in Partitions.

Take for instance 6 letters; a partition into 3's, such as *abc*, *def* contains the 6 duads *ab*, *ac*, *bc*, *de*, *df*, *ef*. A partition into 2's such as *ab.cd.ef* contains the 3 duads *ab*, *cd*, *ef*. Hence if there are α partitions into 3's, and β partitions into 2's, and these contain all the duads each once and only once, $6\alpha + 3\beta = 15$, or $2\alpha + \beta = 5$. The solutions of this last equation are ($\alpha = 0$, $\beta = 5$), ($\alpha = 1$, $\beta = 3$), ($\alpha = 2$, $\beta = 1$), and it is at once seen that the first two sets give solutions of the partition problem, but that the third set gives no solution; thus we have

$\alpha = 0$, $\beta = 5$	$\alpha = 1$, $\beta = 3$
ab . cd . ef	*abc . def*
ac . be . ef	*ad . be . cf*
ad . bf . ce	*ae . bf . cd*
ae . bd . cf	*af . bd . ce.*
af . bc . de	

Similarly for any other number of letters, for instance 15; if we have α partitions into 5's and β partitions into 3's, then if these contain all the duads $4\alpha + 2\beta = 14$, or what is the same $2\alpha + \beta = 7$; if $\alpha = 0$, $\beta = 7$, the partition problem can be solved (this is in fact the problem of the 15 school-girls), but can it be solved for any other values (and if so which values) of α, β? Or again for 30 letters; if we have α partitions into 5's, β partitions into 3's and γ partitions into 2's; then if these contain all the duads $4\alpha + 2\beta + \gamma = 29$; and the question is for what values of α, β, γ, does the partition-problem admit of solution.

The question is important from its connexion with the theory of groups, but it seems to be a very difficult one.

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1879.

therefore

$$D=\frac{[\cos a-\cos(a+nb)]^n-[\cos(a-b)-\cos\{a+(n-1)b\}]^n}{2(1-\cos nb)}.$$

(V). In like manner

$$|\sin a, \sin(a+b)\ldots\sin\{a+(n-1)b\}|$$
$$=\frac{[\sin a-\sin(a+nb)]^n-[\sin(a-b)-\sin\{a+(n-1)b\}]^n}{2(1-\cos nb)}.$$

3. We are also able to solve the partial differential equation

$$\begin{vmatrix} D_1, & D_2, & \ldots D_n \\ D_n, & D_1, & \ldots D_{n-1} \\ \ldots & \ldots & \ldots \\ D_2, & D_3, & \ldots D_1 \end{vmatrix} u=0,$$

where $$D_r=\frac{d}{dx_r}.$$

Viz. the solution of this equation is the sum of the solutions of the different equations

$$\frac{du}{dx_1}+\alpha\frac{du}{dx_2}+\ldots+\alpha^{n-1}\frac{du}{dx_n}=0,$$

where $\alpha^n=1$. This is of Lagrange's form, and

$$u=f(x_2-\alpha x_1, x_3-\alpha^2x_1, \ldots x_n-\alpha^{n-1}x_1).$$

Thus the solution of the given equation is

$$u=\Sigma f(x_2-\alpha x_1, x_3-\alpha^2x_1, \ldots x_n-\alpha^{n-1}x_1),$$

when the summation extends to all values of α being roots of the equation $z^n-1=0$.

ON THE RELATIONS BETWEEN THE ANGLES OF FIVE CIRCLES IN A PLANE OR OF SIX SPHERES IN SPACE

By *Edouard Lucas.*

THE definition of the potency of a point with respect to a circle or a sphere, its expression in the system of Cartesian coordinates, and the most elementary properties of determinants, enable us to arrive immediately at a knowledge of the most general relations concerning the distances of points, angles, straight lines, and circles in a plane, and the angles of planes and spheres in space. The relations of the distances and angles are included, to a great extent, in the following fundamental proposition.

The potencies of any point with respect to five circles in a plane or with respect to six spheres in space are connected by a linear and homogeneous equation in which the sum of the coefficients is zero.

Let $$x_i = x^2 + y^2 - 2a_i x - 2b_i y + c_i,$$
the first member of the equation of a circle in rectangular coordinates; we know that c_i and x_i represent respectively the potency of the origin and of any point M, coordinates x, y, with respect to this circle. Eliminating x, y and x^2+y^2 between the four equations obtained with $i = 1, 2, 3, 4$, we have the identity*

$$\begin{vmatrix} x_1, & a_1, & b_1, & 1 \\ x_2, & a_2, & b_2, & 1 \\ x_3, & a_3, & b_3, & 1 \\ x_4, & a_4, & b_4, & 1 \end{vmatrix} = \begin{vmatrix} c_1, & a_1, & b_1, & 1 \\ c_2, & a_2, & b_2, & 1 \\ c_3, & a_3, & b_3, & 1 \\ c_4, & a_4, & b_4, & 1 \end{vmatrix}.$$

Denote by l, m, n, p the coefficients of x_1, x_2, x_3, x_4 in the development of the determinant forming the first member, and by q the value of the second member, we have then

$$lx_1 + mx_2 + nx_3 + px_4 - q = 0 \;\ldots\ldots\ldots\ldots (1);$$

but if we replace x_1, x_2, x_3, x_4 by unity or by the same number the first determinant vanishes, and we have

$$l + m + n + p = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We thus obtain the following principle:

The potencies of a point with respect to four circles in a plane are connected by a linear equation, not homogeneous, in which the sum of the coefficients of the powers is zero.

If the four circles are orthogonal to a fifth, the centre of the latter has the same potency with respect to the four circles, whence $q = 0$, and consequently:

The potencies of a point with respect to four circles, orthogonal to the same circle, are connected by a linear and homogeneous equation in which the sum of the coefficients is zero.

To prove the fundamental theorem it suffices to consider a fifth circle, and we have, for example,

$$\begin{aligned} l'x_1 + m'x_2 + n'x_3 + p'x_5 - q' &= 0 \ldots\ldots\ldots\ldots (3), \\ l' + m' + n' + p' \qquad &= 0 \ldots\ldots\ldots\ldots (4). \end{aligned}$$

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* *Nouvelles Annales de Mathématiqnes*, Second Series. t. XV. p. 205.

Multiplying the two members of the equalities (1) and (2) by q', those of the equalities (3) and (4) by q, and subtracting, we have the demonstration of the theorem.

We define the *mutual potency* of two circles radii r_i and r_j, centres O_i and O_j, distance between the centres d_{ij}, to be A_{ij}, where

$$2r_ir_jA_{ij} = r_i^2 + r_j^2 - d_{ij}^2 \quad \ldots\ldots\ldots\ldots\ldots (5);$$

when the circles cut, A_{ij} represents the cosine of the angle of the two circles; we have also

$$d_{ii} = 0, \quad A_{ii} = 1, \quad A_{ij} = A_{ji}.$$

Now suppose the point M to be at the centre O_j of a circle of radius r_j; the potency of this point with respect to the circle of centre O_i is

$$x_i = d_{ij}^2 - r_i^2,$$

viz.

$$x_i = r_j^2 - 2r_ir_jA_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

but, by the fundamental theorem,

$$lx_1 + mx_2 + nx_3 + px_4 + qx_5 = 0,$$
$$l + m + n + p + q = 0;$$

and consequently comparing the three preceding formulæ,

$$lr_1r_jA_{1j} + mr_2r_jA_{2j} + nr_3r_jA_{3j} + pr_4r_jA_{4j} + qr_5r_jA_{5j} = 0.$$

Making j equal to 1, 2, 3, 4, 5, in the preceding equation we obtain five equations, which give, by the elimination of l, m, n, p, q, the determinant

$$\begin{vmatrix} r_1r_1A_{11}, & r_1r_2A_{12}, & r_1r_3A_{13}, & r_1r_4A_{14}, & r_1r_5A_{15} \\ r_2r_1A_{21}, & r_2r_2A_{22}, & r_2r_3A_{23}, & r_2r_4A_{24}, & r_2r_5A_{25} \\ r_3r_1A_{31}, & r_3r_2A_{32}, & r_3r_3A_{33}, & r_3r_4A_{34}, & r_3r_5A_{35} \\ r_4r_1A_{41}, & r_4r_2A_{42}, & r_4r_3A_{43}, & r_4r_4A_{44}, & r_4r_5A_{45} \\ r_5r_1A_{51}, & r_5r_2A_{52}, & r_5r_3A_{53}, & r_5r_4A_{54}, & r_5r_5A_{55} \end{vmatrix} = 0 \ldots\ldots (7).$$

If no one of the radii is zero, we can suppress all the radii; if one of them r_i is zero, we can replace $r_ir_jA_{ij}$ by $d^2_{ij} - r_j^2$, and $r_ir_iA_{ii}$ by 0; if two radii r_i and r_j are zero, we replace r_ir_j by d^2_{ij}. Thus, when the system of two circles is formed of a circle and a point, we replace in the relation (7) the mutual potency of the two circles by the potency of the point with regard to the circle; and if the system is formed of two points, we replace the mutual potency by the square of their distance.

If the radius r_i of one of the circles increases indefinitely, and if the circle becomes a straight line, divide by r_i and replace A_{ij} by $\frac{\delta_j}{r_j}$ or by the cosine of the angle between the straight line and the circle, when the straight line and circle cut one another. Thus, when the system of two circles is formed of a circle and a straight line, we replace in the relation (7) the mutual potency by the quotient of the distance of the centre to the straight line by the radius; we see also that when the system of two circles is formed of a point and a straight line, we replace the mutual potency by the distance of the point from the straight line.

In the system formed by a circle and a straight line which passes to an indefinite distance, we replace the mutual potency by the reciprocal of the radius.

In the system formed by a point and the straight line at infinity, we replace the mutual potency by unity; in the system formed by two straight lines we replace the mutual potency by the cosine of their angle; if one of the straight lines passes to an infinite distance, we replace the mutual potency by zero.

Consequently, applying, as has been said, the notion of the mutual potency of the points, of the straight lines, of the circles and of the straight lines at infinity, we have the following theorem:

The determinant formed by the mutual potencies of any points, straight lines or circles in a plane, whose number is at least equal to *five*, is identically zero.

We have evidently the same theorem in space for any six points, planes or spheres, or for any one or a greater number of elements.

The following are some corollaries from this important theorem:

1. If the circle O_5 is a point on the circle O_4, we have the formula

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14}, & \frac{x_1}{r_1} \\ A_{21}, & A_{22}, & A_{23}, & A_{24}, & \frac{x_2}{r_2} \\ A_{31}, & A_{32}, & A_{33}, & A_{34}, & \frac{x_3}{r_3} \\ A_{41}, & A_{42}, & A_{43}, & A_{44}, & 0 \\ \frac{x_1}{r_1}, & \frac{x_2}{r_2}, & \frac{x_3}{r_3}, & 0, & 1 \end{vmatrix} = 0.$$

This is the equation of the system of two circles cutting three given circles $x_1 = 0$; $x_2 = 0$, $x_3 = 0$ at given angles.

2. We obtain, besides, the four groups of these circles conformably to the rule of signs given in the *Nouvelle Correspondance Mathématique.*

3. If $A_{14} = A_{24} = A_{34} = \pm 1$, we have the equation of the system of two circles touching internally or externally three given circles.

4. If $A_{14} = A_{24} = A_{34} = 0$, we obtain the equation of the circle orthogonal to three given circles.

5. If the given circles reduce to three points, we obtain the equation of the circumscribed circle.

6. If the circles become three straight lines, we obtain the equation of the inscribed and escribed circles.

7. If, besides, $A_{14} = A_{24} = A_{34} = 0$, we obtain the equation of the double straight line at infinity.

8. Suppose now that the circle O_5 becomes the straight line at infinity, we then have the formula

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14}, & \frac{1}{r_1} \\ A_{21}, & A_{22}, & A_{23}, & A_{24}, & \frac{1}{r_2} \\ A_{31}, & A_{32}, & A_{33}, & A_{34}, & \frac{1}{r_3} \\ A_{41}, & A_{42}, & A_{43}, & A_{44}, & \frac{1}{r_4} \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & 0 \end{vmatrix} = 0;$$

this is the equation giving the radii of the three circles which cut three given straight lines or circles at given angles.

9. If $A_{14} = A_{24} = A_{34} = \pm 1$, we find again the formula giving the radii of the system of circles touching three given circles. (Bauer, *Journal de Schlömilch*, t. v.).

10. If $A_{14} = A_{24} = A_{34} = 0$, we find the radius of the circle orthogonal to three given circles.

11. If $r_1 = r_2 = r_3 = 1$, we obtain the radius of the circle circumscribing the triangle, given by the three points O_1, O_2, O_3, and consequently the area of the triangle.

12. If $r_1 = r_2 = r_3 = r_4 = 1$, we obtain the relation given by Cayley between the distances of four points in a plane (*Cambridge Journal*, t. II.).

13. If the elements O_4 and O_5 become two straight lines at infinity, we obtain the relation between the cosines of the angles of three directions in a plane.

14. If the elements O_1, O_2, O_3 represent the origin of the coordinates and the two axes, O_4 any circle whatever, and O_5 a point on this circle, we obtain the equation of the circle O_4 in oblique coordinates.

15. If the element O_5 is replaced by the straight line at infinity, we obtain the radius of a circle with given centre, which cuts at given angles the two axes of coordinates, &c.

These considerations are also applicable very easily to space.

Paris, *December*, 1877.

MATHEMATICAL NOTES.

Proof of the Theorem in Kinematics, vol. VII. p. 190.

Let P, P' be two points on the moving plane, and (P), (P') the areas described by them.

Let $PP' = r$, and let PP' make N revolutions.

Let n be the total movement of P' perpendicular to $P'P$.

Then $(P) - (P') = nr + N\pi r^2$.

Take P' as origin and the direction of $P'P$ in which n is a maximum $(= n')$ as initial line.

Then $$n = n' \cos\theta.$$

Thus $$(P) - (P') = n'r\cos\theta + N\pi r^2,$$

the equation to a family of concentric circles.

Transform to centre, then

$$(P) = N\pi\,(r^2 - a^2)$$

where a = radius of circle corresponding to $(P) = 0$.

A. B. KEMPE.

Fluid Motion in a Rotating Semicircular Cylinder.

It may be useful to put on record the solution of this simple case of fluid motion. Take the centre of the semicircle as origin and one of the bounding radii as initial line.

MATHEMATICAL NOTES.

On Long Successions of Composite Numbers.

Mr. Glaisher has given in the *Messenger* for November, 1877, and March, 1878, interesting results upon the sequences of composite numbers in the natural series. To the explanations given by the author, I add the following reflections upon the appearance of sequences much earlier than the theory indicates.

Denote by $N, = 2n+1$, any uneven number, by $P(q)$ the product of all the primes $2.3.5\ldots 9$, q being the greatest prime inferior to n, and by $S_n(x)$ the series of the N consecutive numbers $xP(q)-n, \ldots xP(q), \ldots xP(q)+n$.

If we suppose $1<\alpha<n+1$, the number $xP(q)\pm\alpha$ is evidently composite, whatever integer value x may have.

Therefore, if $xP(q)-1$ and $xP(q)+1$ are composite numbers, the series $S_n(x)$ will be formed of N composite numbers. We have, for example, for $x=1$,

$$P(17)-1=61\times 8369, \qquad P(17)+1=19\times 97\times 277,$$

$$P(19)-1=53\times 197\times 929, \quad P(19)+1=347\times 27953,$$

whence the series

$$510492, \ldots\ 510510, \ldots\ 510528$$

contains 37 composite numbers, and the series

$$9699668, \ldots\ 9699690, \ldots\ 9699712$$

contains 45 consecutive composite numbers.

Thus, again, the series $S_{10}(x)$ contains 21 composite numbers for the values of x,

8, 15, 25, 26, 31, 33, 34, 37, 38, 45, 52, 54, 56, 58, 62, 71, 77, 79, 80, 82, 84, 91, 98, ...

In the general case, where q is given, we determine x by solving the two simultaneous congruences

$$xP(q)\equiv 1 \pmod{g} \quad\text{and}\quad xP(q)\equiv -1 \pmod{h},$$

g and h denoting the two primes that follow q.

EDOUARD LUCAS.

Paris, *August*, 1878.